SOLUTIONS OF PROBLEMS.

245 (Number Theory) [May, 1916; May, 1917]. Proposed by NORMAN ANNING, Chilliwack, B. C.

Show that $x^2 + y^2 = (a_1a_2 \cdots a_m)^n$ has $4(n+1)^m$ solutions in integers, in 2^{m+2} of which x and y are relatively prime, the a's being primes of the form $4k+1$ and n a positive integer.

I. Solution by Frank Irwin, University of California.

In Dirichlet's *Zahlentheorie* (4th edition, page 164) it is shown that the equation $x^2 + y^2 = M$ has 2^{m+2} solutions in integers x, y relatively prime, where M has m different prime factors (all of them being of the form $4k+1$). It may be expressly noted that the solutions, if any, in which x or y are zero are not counted in the number 2^{m+2}.

The solutions of our given equation in which x, y are not relatively prime may evidently be obtained by taking relatively prime solutions of equations of the form $x'^2 + y'^2 = N$, where N is the quotient of the right side of our equation by any square integer d^2; then $x = dx'$, $y = dy'$. The number of these solutions may be found as follows:

(i) n odd. N may contain a_i 1, 3, $\cdots$ or n times, $i = 1, 2, \cdots m$, and for each such choice of N we have 2^{m+2} solutions; altogether, then, $[(n+1)/2]^m 2^{m+2} = 4(n+1)^m$ solutions.

(ii) n even. Here N need not contain all the m a's as factors, and we must consider separately the cases thus arising. If N has m different prime factors, we get $(n/2)^m 2^{m+2}$ solutions; if N has $m-1$ different prime factors, we get $m(n/2)^{m-1}2^{m+1}$ solutions; if N has $m-2$ different prime factors, we get $\binom{m}{2}(n/2)^{m-2}2^m$ solutions; etc., etc.; that is, in all

$$4\left[n^m + mn^{m-1} + \binom{m}{2}n^{m-2} + \cdots + 1\right] = 4(n+1)^m$$

solutions.

II. Solution by C. F. Gummer, Queen's University.

It is well known (see, for instance, Barlow's *Theory of Numbers*, Chapter IX) that for every prime a_i of the form $4k+1$ positive integers x_i, y_i, relatively prime, may be found such that $x_i^2 + y_i^2 = a_i$. Let the even one be x_i, and let us write $x_i = \sqrt{a_i}\cos\theta_i$, $y_i = \sqrt{a_i}\sin\theta_i$. Then we can prove that a solution in integers of

$$x^2 + y^2 = a_1^{n_1}a_2^{n_2}\cdots a_m^{n_m} = A, \tag{1}$$

where the a's are primes of the form $4k+1$, is given by

$$x = \sqrt{A}\cos\Theta, \qquad y = \sqrt{A}\sin\Theta, \qquad \Theta = k\pi/2 + \sum_i r_i\theta_i, \tag{2}$$

where r_i is any one of the integers n_i, $n_i - 2$, $n_i - 4$, $\cdots$, $-n_i$, and k is an integer.

There are no solutions of (1) other than those included in (2). For let $x^2+y^2=x'^2+y'^2=B$, where B is odd, x, y, x', y' positive or zero, x and x' even, and $x > x'$. Since

$$\tfrac{1}{2}(x+x')\cdot\tfrac{1}{2}(x-x') = \tfrac{1}{2}(y'+y)\cdot\tfrac{1}{2}(y'-y),$$

and neither member is zero, we infer that every divisor of $\frac{1}{2}(x+x')$ is a divisor of either $\frac{1}{2}(y'+y)$ or $\frac{1}{2}(y'-y)$, and so for $\frac{1}{2}(x-x')$. Hence, we may write

$$\tfrac{1}{2}(x+x') = pr, \qquad \tfrac{1}{2}(x-x') = qs, \qquad \tfrac{1}{2}(y'+y) = ps, \qquad \tfrac{1}{2}(y'-y) = qr,$$

and consequently

$$x = pr + qs, \qquad y = ps - qr, \qquad x' = pr - qs, \qquad y' = ps + qr,$$

and

$$B = x^2 + y^2 = (p^2+q^2)(r^2+s^2).$$

Since no one of p, q, r, s can be zero, it follows that $x^2 + y^2 = B$ can admit two essentially distinct solutions only if B is composite, and therefore that the θ_i are unique. Also, on writing

$$p = \sqrt{C}\cos\theta, \qquad q = \sqrt{C}\sin\theta, \qquad r = \sqrt{D}\cos\phi, \qquad s = \sqrt{D}\sin\phi,$$

we see that if $x^2 + y^2 = B$ has two such solutions they are deducible from integral solutions of

$x^2 + y^2 = C$ and $x^2 + y^2 = D$ $(CD = B)$ by the rule $x = \sqrt{B} \cos (\theta \mp \phi)$, $y = \sqrt{B} \sin (\theta \mp \phi)$. If we allow for the interchange of x and y and for negative values, the rule becomes

$$x = \sqrt{B} \cos (l\pi/2 \pm \theta \pm \phi), \qquad y = \sqrt{B} \sin (l\pi/2 \pm \theta \pm \phi);$$

and on applying this method successively to A and its divisors we get the formula (2).

We wish to know how many distinct solutions are given by (2). Two solutions given by

$$\Theta' = k'\pi/2 + \sum_i r_i'\theta_i \qquad \text{and} \qquad \Theta'' = k''\pi/2 + \sum_i r_i''\theta_i$$

are equal only if

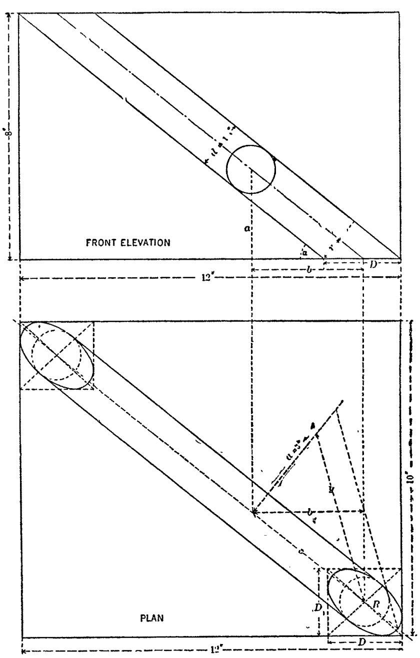
$$(k' - k'')\pi/2 + \sum_i (r_i' - r_i'')\theta_i$$

is a multiple of 2π. If this is so and $r_1' - r_1'' \neq 0$, it follows that

$$\cos (r_1' - r_1'')\theta_1 = \cos \left\{ K\pi/2 + \sum_{i=2}^{m} (r_i' - r_i'')\theta_i \right\}.$$

But the left member, being a polynomial of degree $|r_1' - r_1''|$ in $\cos\theta$ with integral coefficients of which the highest is a power of 2, is a fraction with denominator $a_i^{|r_1'-r_1''|}$, while the right member cannot be of this form. Hence $r_1' = r_1''$. Two solutions are then identical only if $r_i' = r_i''$ $(i = 1, \cdots, m)$ and $k' - k''$ is a multiple of 4. The number of effectively distinct combinations of values of $k, r_1, \cdots, r_n$ is therefore $4(n_1 + 1)(n_2 + 1) \cdots (n_m + 1)$.

In some of the solutions given in (2), x and y will admit the common divisor $E = a_1^{s_1}a_2^{s_2}\cdots a_m^{s_m}$. These are found by multiplying by E the roots of the equation $x^2 + y^2 = a_1^{n_1-2s_1}a_2^{n_2-2s_2}\cdots$, and therefore occur in the cases where $|r_i| \leqq n_i - 2s_i$ $(i = 1, 2, \cdots m)$. The solutions have no common divisor when there is no number of the form E other than unity for which this condition holds, that is when $r_i = \pm n_i$ $(i = 1, \cdots m)$. The number of relatively prime solutions is therefore $4.2^m = 2^{m+2}$.



2696 [April, 1918]. **Proposed by L. E. LUNN, Heron Lake, Minnesota.**

An air pipe 18 inches in diameter passes diagonally through a room from one lower corner to the opposite upper corner leaving through elliptical openings in the floor and ceiling, so that the ellipses are tangent to two boundaries of the floor and to the two opposite boundaries of the ceiling. If the room is $10 \times 12 \times 8$ feet, find the remaining cubic capacity of the room.

SOLUTION BY A. R. NAUER, St. Louis, Missouri.

Make D the apparent width of the floor or ceiling contact as seen at the front elevation, and D_1 the same for the side elevation. Then

$$D = d/\sin\alpha = -\frac{27}{61.75} + \frac{\sqrt{468 \times 61.75 + 27^2}}{61.75} = 2.35025 +$$

and

$$D_1 = -\frac{22.5}{61.75} + \frac{\sqrt{369 \times 61.75 + 22.5^2}}{61.75} = 2.10715 +$$

a is the length of a perpendicular to the floor from an arbitrary point A on the center line of the pipe; a is here taken 3 feet.

r is the radius and d the diameter of the pipe.

R is the half major axis of the bounding ellipses, and

b is the distance, in front elevation from the foot of perpendicular a, to center of D, which is also the center of contact ellipses;